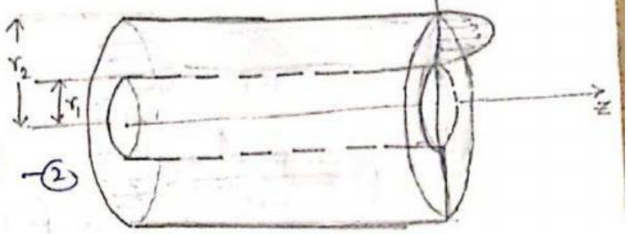
Flow between Two Coaxial Cylinders:

Assuming all the Conditions of flow are same as in Hagen Poi. while the only difference are the Boundary Conditions,

Let $V_z(r_1) = V_z(r_2) = 0$ — ①

Consider the solution of Eqn of motion



$$V_z = \left(\frac{1}{4\mu}\frac{dp}{dz}\right)\left(r^2 + A.\log r + B\right) \quad —②$$

using B.C. (1)

$$\cancel{\frac{1}{4\mu}\frac{dp}{dz}}\left(r_1^2 + A\log r_1 + B\right) = 0 \quad — 3(i)$$

& $$\cancel{\frac{1}{4\mu}\frac{dp}{dz}}\left(r_2^2 + A\log r_2 + B\right) = 0 \quad — 3(ii)$$

3(ii) − 3(i)

$$A \log\left(\frac{r_2}{r_1}\right) = -\cancel{\frac{1}{4\mu}\frac{dp}{dz}}\left(r_2^2 - r_1^2\right)$$

$$A = \cancel{\frac{1}{4\mu}\frac{dp}{dz}}\frac{-(r_2^2 - r_1^2)}{\log\left(\frac{r_2}{r_1}\right)} = \cancel{\frac{1}{4\mu}\frac{dp}{dz}}\frac{-r_1^2(n^2-1)}{\log n} \quad ; \; n = \frac{r_2}{r_1}$$

~~$B = \frac{1}{4\mu}\frac{dp}{dz} r_1^2 - A\log r_1 = \frac{1}{4\mu}\frac{dp}{dz} r_1^2 - \frac{1}{4\mu}\frac{dp}{dz}\frac{r_1^2(n^2-1)}{\log n}\log r_1$~~

$$B = -r_1^2 - A\log r_1 = -r_1^2 + \frac{r_1^2(n^2-1)}{\log n}\log r_1$$

So, with these value of A & B

$$V_z = \frac{1}{4\mu}\frac{dp}{dz}\left[r^2 - \frac{r_1^2(n^2-1)}{\log n}\log r - r_1^2 + \frac{r_1^2(n^2-1)}{\log n}\log r_1\right]$$

$$= \frac{1}{4\mu}\frac{dp}{dz}\left[(r^2 - r_1^2) - \frac{r_1^2(n^2-1)}{\log n}\cdot\log\left(\frac{r}{r_1}\right)\right]$$

$$V_z = \frac{-1}{4\mu}\frac{dp}{dz}\left[(r_1^2 - r^2) + \frac{(n^2-1)}{\log n}r_1^2\cdot\log\left(\frac{r}{r_1}\right)\right]$$

Volumetric flow

$$Q = \int_0^{2\pi} \int_{r_1}^{n r_1} V_z . r\,dr\,d\theta$$

$$= -\frac{1}{4\mu}\frac{dp}{dz} \cdot 2\pi \int_{r_1}^{n r_1} r \left[(r_1^2 - r^2) + \frac{(n^2-1)}{\log n} r_1^2 . \log\left(\frac{r}{r_1}\right)\right] dr$$

$$= \frac{-\pi}{2\mu}\frac{dp}{dz} \int_{r_1}^{n r_1} \left[r_1^2 . r - r^3 + \frac{(n^2-1)}{\log n} r_1^2 . r . \log\left(\frac{r}{r_1}\right)\right] dr$$

$$= \frac{-\pi}{2\mu}\frac{dp}{dz} \left[\frac{r_1^2 r^2}{2} - \frac{r^4}{4} + \frac{(n^2-1)}{\log n} r_1^2 \frac{r^2}{2} \cdot \left(\log\frac{r}{r_1} - \frac{1}{2}\right)\right]_{r_1}^{n r_1}$$

$$= \frac{-\pi}{2\mu}\frac{dp}{dz} \Big[\frac{n^2 r_1^4}{2} - \frac{n^4 r_1^4}{4} + \frac{(n^2-1)}{\log n} \frac{n^2 r_1^4}{2}\left(\log n - \frac{1}{2}\right)$$

$$- \frac{r_1^4}{2} + \frac{r_1^4}{4} - \frac{(n^2-1)}{\log n}\frac{r_1^4}{2}\left(\log 1 - \frac{1}{2}\right)\Big]$$

$$= \frac{-\pi}{2\mu}\frac{dp}{dz} . r_1^4 \left[\frac{n^2}{2} - \frac{n^4}{4} + \frac{(n^2-1)}{\log n} . \frac{n^2}{2}\left(\log n - \frac{1}{2}\right) - \frac{1}{2} + \frac{1}{4} + \frac{(n^2-1)}{4\log n}\right]$$

$$= \frac{-\pi}{2\mu}\frac{dp}{dz} . r_1^4 \left[\frac{n^2}{4} + \frac{n^2}{4}(n^2-1) - \frac{n^2}{4}\frac{(n^2-1)}{\log n} - \frac{1}{4} + \frac{(n^2-1)}{4\log n}\right] ✓$$

$$= \frac{-\pi}{2\mu}\frac{dp}{dz} \cdot r_1^4 \left[\frac{n^4}{4} - \frac{1}{4} - \frac{(n^2-1)^2}{4\log n}\right]$$

$$Q = \frac{-\pi}{8\mu}\frac{dp}{dz} . r_1^4 . \left[(n^4-1) - \frac{(n^2-1)^2}{\log n}\right]$$

$$(V_z)_{avg} = \frac{Q}{\pi(r_2^2 - r_1^2)} = \frac{Q}{\pi r_1^2 (n^2-1)}$$

$$(\tau_{rz}) = \mu \frac{dV_z}{dr} = \mu \left(\frac{-1}{4\mu}\frac{dp}{dz}\right) \cdot \left[-2r + \frac{(n^2-1)}{\log n}\frac{r_1^2}{r}\right]$$

$$(\tau_{rz})_{r=r_1} = \left(-\frac{1}{4}\frac{dp}{dz}\right)\left[-2r_1 + \frac{(n^2-1)}{\log n} r_1\right] = -\frac{r_1}{4}\frac{dp}{dz}\left(\frac{n^2-1}{\log n} - 2\right)$$

$$(\tau_{rz})_{r=r_2} = \left(-\frac{1}{4}\frac{dp}{dz}\right)\left[-2r_2 + \frac{n^2-1}{\log n}\frac{r_1^2}{r_2}\right] = -\frac{1}{4}\frac{dp}{dz}\left[-2nr_1 + \frac{(n^2-1)}{\log n}\frac{r_1}{n}\right]$$

For $(V_z)_{max}$ Consider

$$\frac{dV_z}{dr} = -\frac{1}{4\mu}\frac{dp}{dz}\left[-2r + \frac{(n^2-1)}{\log n}\frac{r_1^2}{r}\right] = 0$$

$$-2r + \frac{(n^2-1)}{\log n}\left(\frac{r_1^2}{r}\right) = 0$$

$$r = \pm r_1\left[\frac{(n^2-1)}{2\log n}\right]^{\frac{1}{2}}$$

This is the radial location of the maximum value of V_z.

we can obtain.

$$(V_z)_{max} = \frac{r_1^2}{4\mu}\frac{dp}{dz}\left[1 - \frac{(n^2-1)}{2\log n}\left\{1 - \log\left(\frac{n^2-1}{2\log n}\right)\right\}\right]$$

Note!

Hagen-Poiseuille flow in straight pipe can be obtained by taking limit $r_1 \to 0$ in expression for V_z. Poiseuille flow between parallel plates which are stationary can be obtained by taking limit $r_1 \to \infty$ such that $r_2 - r_1 =$ Constant. This can be done by taking $r_2 = r_1 + k$ and $r = r_1 + \delta$ and then $r_1 \to \infty$.