Subject :- Dynamics of Machines
Course :- B.Tech
Year :- Third
Branch :- Mechanical Engineering
Faculty :- Er. Sandeep Kumar Gupta
Unit No :- V
Topic :- Mechanical Vibration

Vibration :- A body is said to be vibrate if it has a to and fro motion.

Vibration

- Natural Vibration
  - No friction
  - No External force
- Damped Vibration
  - friction
  - Gradually decreases
- Forced Vibration
  - Repeated force acts continuously

Types of Vibrations : (According to the motion)

(i) Longitudinal Vibrations :-
* If the shaft is elongated and shortened so that the mass moves up and down resulting in tensile and compressive stresses in the shaft.

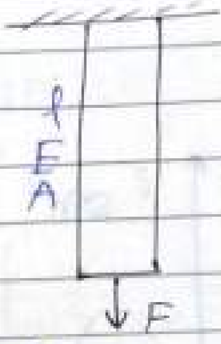
(ii) Transverse Vibration :- When the shaft is bent alternately and tensile and compressive stresses due to bending result, the Vibrations called transverse.

Natural Vibrations

"The vibrations in which there is no friction at all as well as there is no unbalance force after the initial release of the system are known as natural vibrations."

Stiffness - Resistance of material against the deformation.

Longitudinal Stiffness (S)



Force required to produce unit deflection.
deflection → change in length

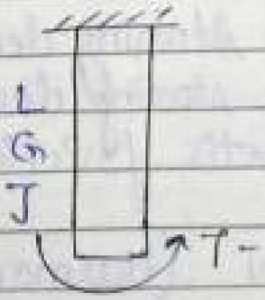
$$E = \frac{F/A}{\Delta l/L} = \frac{FL}{A\Delta L} \Rightarrow F = \frac{AE\Delta L}{L}$$

$$\boxed{S = \frac{AE}{L}}$$

Torsional stiffness (S_θ)

Torque required to produce unit angle of twist.

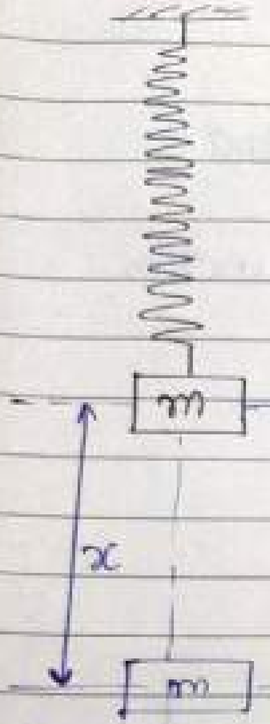
Angle of Twist:



Twisting moment

$$\theta = \frac{TL}{GJ} \Rightarrow \boxed{T = \frac{GJ\theta}{L}}$$

$$\boxed{S_\theta = \frac{GJ}{L}}$$



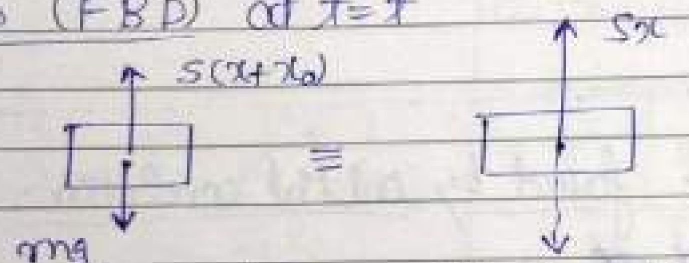
Equilibrium equation

$$\boxed{mg = Sx_0}$$

Mean or equi^m position

At $(t = t)$

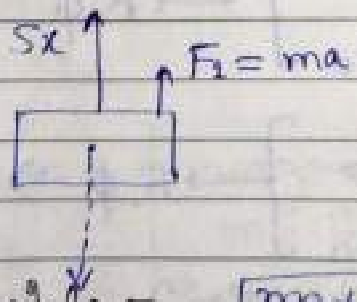
Mass (FBD) at $t = t$



Newton second law - (In the dir^n of disturbance)

$$0 - Sx = ma \Rightarrow \boxed{ma + Sx = 0}$$

D'Alembert



D'Alembert principle - $\boxed{ma + Sx = 0}$

$$ma + Sx = 0 \qquad (a = \dot{v} = \ddot{x})$$

$$m \cdot \ddot{x} + Sx = 0$$

$$\Rightarrow \boxed{\ddot{x} + \frac{S}{m} x = 0} \leftarrow \text{Equa}^n \text{ of natural system}$$

The solution of above equa.n

$x = R \sin\left[\sqrt{\frac{S}{m}}\, t + \phi\right]$ where $R, \phi \rightarrow$ constant

It indicates a harmonic function (vibration)

Amplitude is constant

With frequency - $\omega_n = \sqrt{\frac{S}{m}}$ rad/sec

$T = \frac{2\pi}{\omega_n}$ (sec) so $f_n = \frac{1}{2\pi}\sqrt{\frac{S}{m}} = \frac{\omega_n}{2\pi}$

$$\boxed{f_n = \frac{\omega_n}{2\pi}}$$

$R, \phi \Rightarrow$ constant can be found by initial conditions

(1) at $t=0$ $\begin{cases} x = x_0 \\ \dot{x} = 0 \end{cases} \longrightarrow R, \phi \Rightarrow ??$

(2) At $t=0$ $\begin{cases} x = 0 \\ \dot{x} = v_0 \end{cases} \longrightarrow R, \phi \Rightarrow ??$

(3) At $t=0$ $\begin{cases} x = x_0 \\ \dot{x} = v_0 \end{cases} \longrightarrow R, \phi \Rightarrow ??$

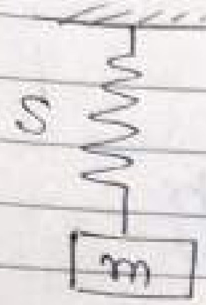
Find equa.n of natural vibration?

$$\boxed{\ddot{x} + (\omega_n)^2 \cdot x = 0}$$

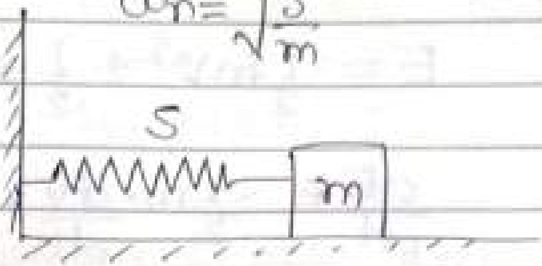
Ques. $5\ddot{x} + 3x = 0$, $\omega_n = ?$

$$\ddot{x} + \frac{3}{5}x = 0$$

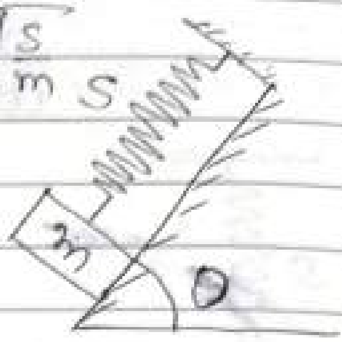
$$\text{So } \omega_n = \sqrt{\frac{3}{5}} \text{ rad/sec.}$$



$\omega_n = \sqrt{\frac{s}{m}}$



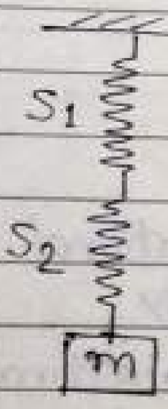
$\omega_n = \sqrt{\frac{s}{m}}$



$\omega_n = \sqrt{\frac{s}{m}}$

Combination of spring

(1) Series

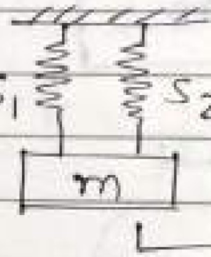


$$\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2}$$

$$s = \frac{s_1 s_2}{s_1 + s_2}$$

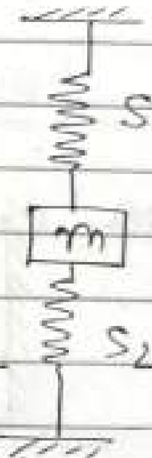
$$\omega_n = \sqrt{s/m}$$

(2) Parallel



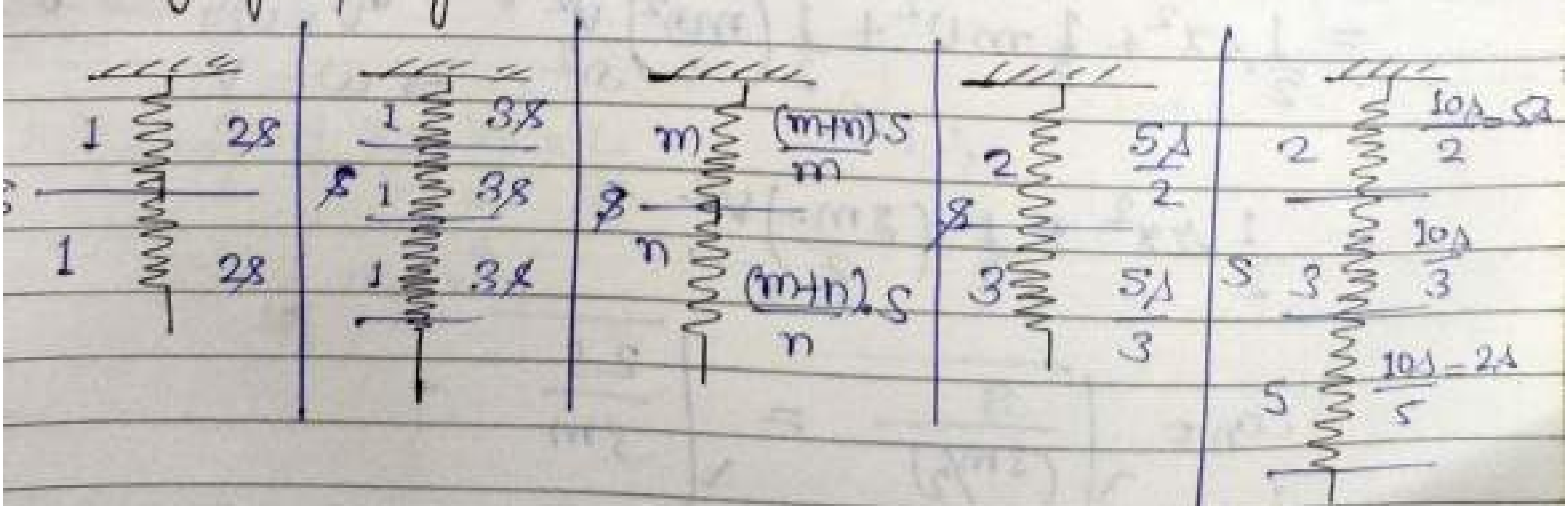
$$s = s_1 + s_2$$

$$\omega_n = \sqrt{s/m}$$



(also parallel connection)

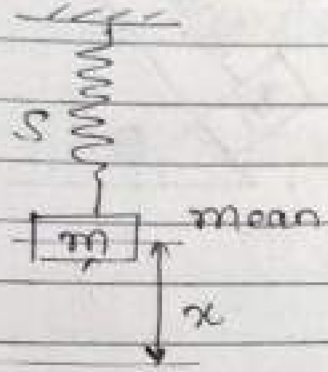
Cutting of springs



Energy method

$E = \text{constant}$

$\frac{dE}{dt} = 0$



$$E = \frac{1}{2}mv^2 + \frac{1}{2}sx^2$$

$$\frac{dE}{dt} = \frac{1}{2}m \cdot \frac{dv}{dt} \times 2v + \frac{1}{2}s \times 2x \cdot \frac{dx}{dt}$$

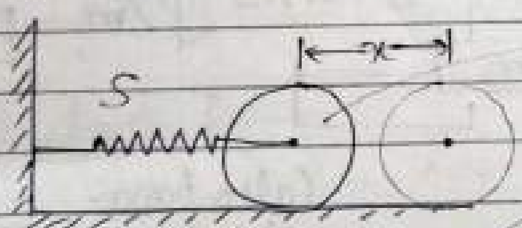
but $\frac{dE}{dt} = 0$ so

$$ma + sx = 0 \Rightarrow \boxed{\ddot{x} + \frac{s}{m} \cdot x = 0}$$

$$\boxed{\omega_n = \sqrt{s/m}}$$

Que

Disc M, R



Disc translational energy

~~Afoo~~ ~~Ro~~ disc rotational energy

$$E = \frac{1}{2}sx^2 + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

for pure rolling $v = \omega r$

$\Rightarrow \omega = \frac{v}{r}$

$$= \frac{1}{2}sx^2 + \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{mr^2}{2}\right)\frac{v^2}{r^2}$$

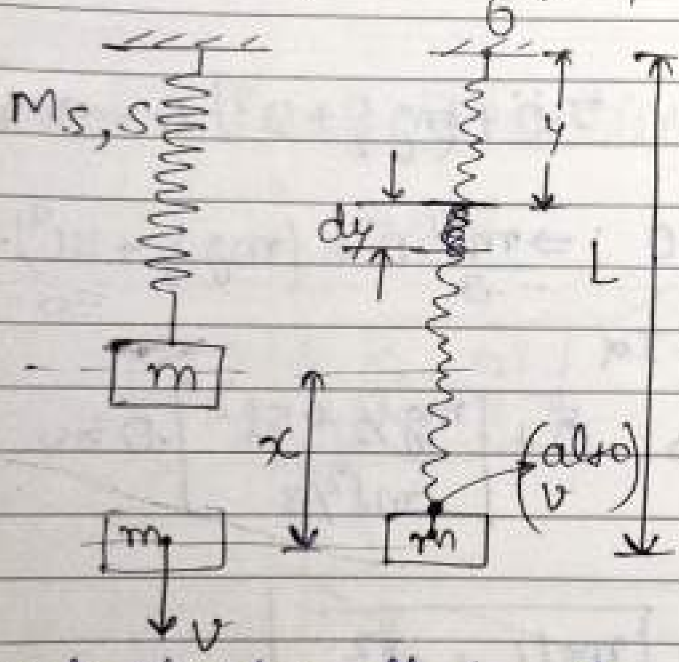
$$= \frac{1}{2}sx^2 + \frac{1}{2}\left(\frac{3m}{2}\right)v^2$$

$$\omega_n = \sqrt{\frac{s}{(3m/2)}} = \sqrt{\frac{2s}{3m}}$$

Moment of inertia

1) Ring $\longrightarrow$ $MR^2 = I$
   Hollow cylinder $\longrightarrow$ $MR^2 = I$
2) Disc $\longrightarrow$ $(MR^2)/2 = I$
   Solid cylinder $\longrightarrow$ $(MR^2)/2 = I$
3) Hollow sphere $\longrightarrow$ $(2MR^2/3) = I$
4) Solid Sphere $\longrightarrow$ $\frac{2}{5}MR^2 = I$

Note - If mass of spring is also given



$$K.E._{spring} = \int_0^L \frac{1}{2}\left(\frac{Ms}{L}\cdot dy\right)\left(\frac{v}{L}y\right)^2$$

$$E = \frac{1}{2}Sx^2 + \frac{1}{2}mv^2 + \frac{1}{6}m_sv^2$$

$$E = \frac{1}{2}Sx^2 + \frac{1}{2}\left(m + \frac{m_s}{3}\right)v^2$$

$$\omega_n = \sqrt{\frac{S}{m + \frac{m_s}{3}}} \ rad/sec$$

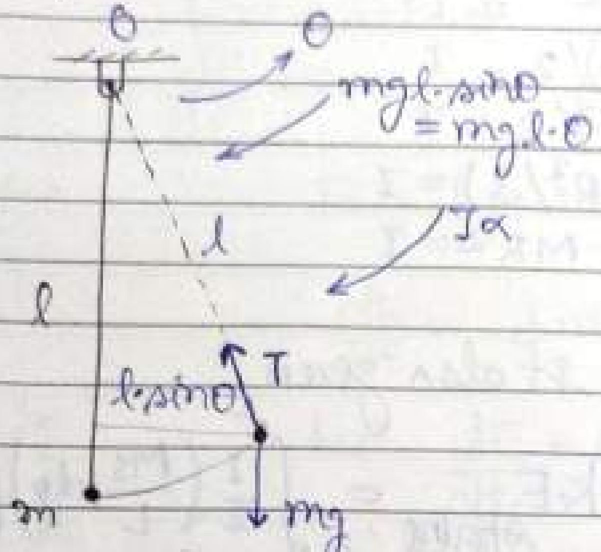
Velocity of small element

$$u = \frac{v}{L}\cdot y$$

$$K.E_{spring} = \frac{1}{2}\frac{m_s}{L}\cdot\frac{v^2}{L^2}\times\frac{L}{3}$$

$$= \frac{1}{6}m_s\cdot v^2$$

Torque Method



$$I\alpha + mg.l.\theta = 0$$

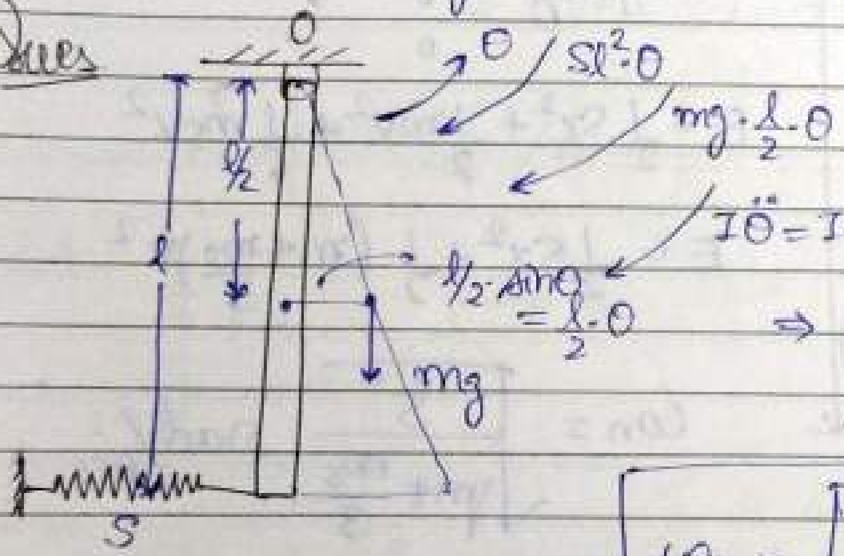
$$\Rightarrow I.\ddot{\theta} + mg.l.\theta = 0$$

$$\Rightarrow ml^2\ddot{\theta} + mg\,l.\theta = 0$$

$$\Rightarrow \ddot{\theta} + \frac{g}{l}.\theta = 0$$

$$\omega_n^2 = g/l \Rightarrow \omega_n = \sqrt{g/l}$$

Ques



$$I.\ddot{\theta} + \left(mg\frac{l}{2} + sl^2\right).\theta = 0$$

$$\Rightarrow \frac{ml^2}{3}.\ddot{\theta} + \left(mg\frac{l}{2} + sl^2\right).\theta = 0$$

$$\Rightarrow \ddot{\theta} + \left[\frac{mg\,l/2 + sl^2}{ml^2/3}\right].\theta = 0$$

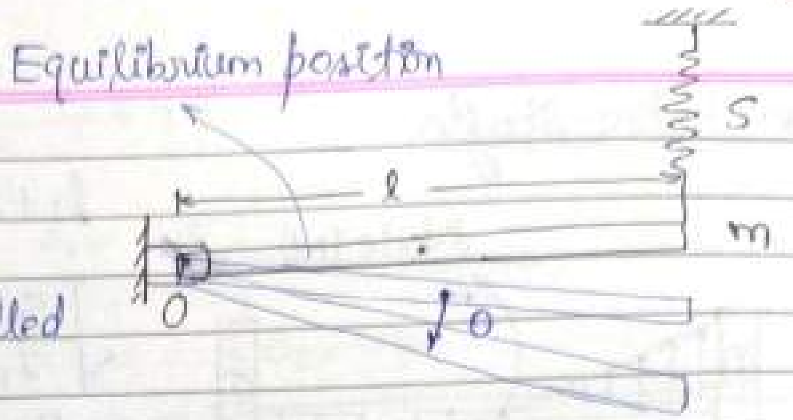
$$\boxed{\omega_n = \sqrt{\frac{mg\,l/2 + sl^2}{ml^2/3}}}$$

$$I = \frac{ml^2}{12}$$

but $I = \frac{ml^2}{12} + m\left(\frac{l}{2}\right)^2$

$$= \frac{ml^2}{3}$$

Ques



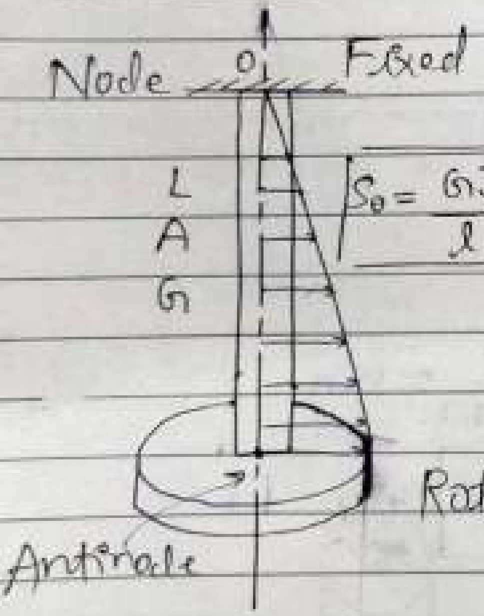
$mg \times \frac{l}{2} = S \cdot x_0 \times l$

mg torque is cancelled by Sx_0 torque

mg torque will not be taken

$$\omega_n = \sqrt{\frac{SL^2}{ml^2/3}} = \sqrt{\frac{3S}{m}}$$

Torsional vibrations:



$\boxed{S_\theta = \frac{GJ}{l}}$

θ (Angle of twist)

$S_\theta \cdot \theta$

$I\ddot{\theta}$

$I\ddot{\theta} + S_\theta \cdot \theta = 0$

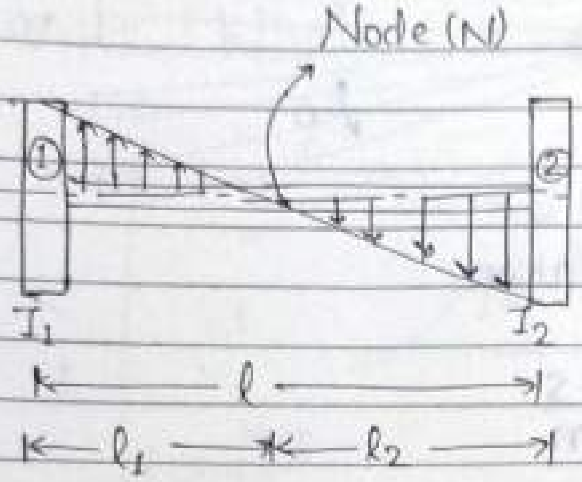
$\Rightarrow \ddot{\theta} + \left(\frac{S_\theta}{I}\right) \cdot \theta = 0$

$$\boxed{\omega_n = \sqrt{\frac{S_\theta}{I}} \text{ rad/sec}}$$

If shaft is also having M.I. → I_{shafts}

$$\boxed{\omega_n = \sqrt{\frac{S_\theta}{I + \frac{I_{shaft}}{3}}}}$$

Two Rotor shafts



$$l_1 + l_2 = l \quad \text{——— (1)}$$

$$\sqrt{\frac{S_{O1}}{I_1}} = \sqrt{\frac{S_{O2}}{I_2}}$$

$$\Rightarrow \frac{S_{O1}}{I_1} = \frac{S_{O2}}{I_2}$$

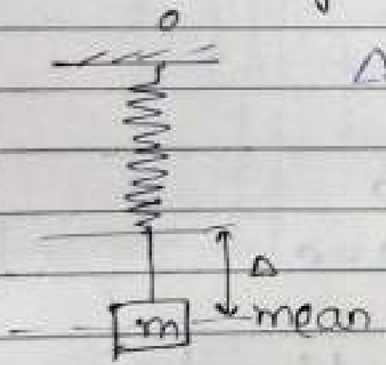
$$\Rightarrow \frac{GJ}{l_1 I_1} = \frac{GJ}{l_2 I_2}$$

$$\Rightarrow \frac{l_1}{l_2} = \frac{I_2}{I_1} \quad \text{——— (2)}$$

from equa.^{n} (1) and (2) $l_1 = ?$ and $l_2 = ?$

Rayleigh's method

(Method of static deflection) (Δ)

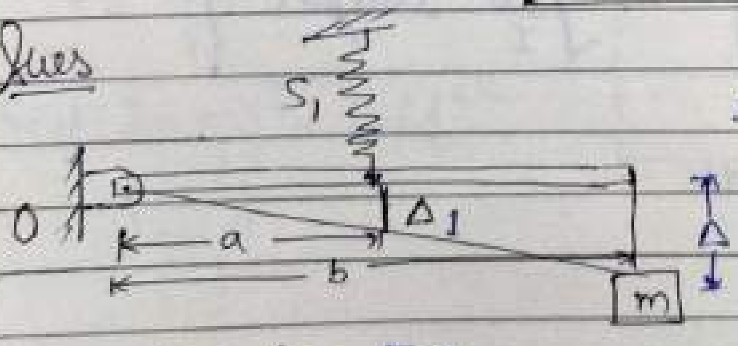


$$\Delta = \frac{mg}{S}$$

$$\sqrt{\frac{g}{\Delta}} = \sqrt{\frac{g}{mg/S}} = \sqrt{\frac{S}{m}} = \omega_n$$

$$\boxed{\omega_n = \sqrt{\frac{g}{\Delta}} = \sqrt{\frac{S}{m}}} \; *$$

Ques



$$\Delta_1 = \frac{mg(b/a)}{S_1}$$

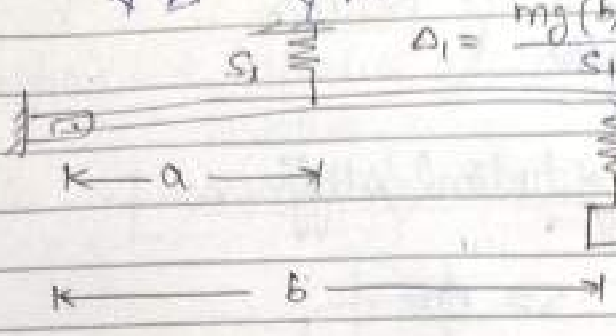
$$mg.b = F.a$$

$$\Rightarrow F = mg.b/a$$

$$\frac{\Delta}{b} = \frac{\Delta_1}{a} \Rightarrow \Delta = \Delta_1 \cdot \frac{b}{a} \Rightarrow \Delta = \frac{mg(b/a)^2}{S_1}$$

$\omega_n = \sqrt{\frac{g}{\Delta}} = \sqrt{\frac{S}{m}} \rightarrow ??$

Ques

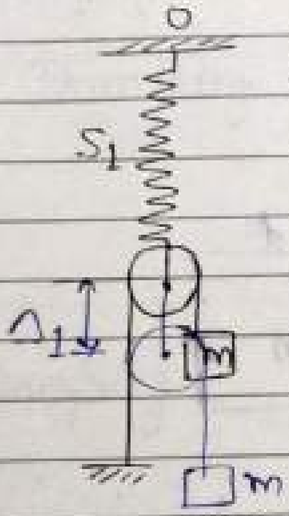


$\Delta_1 = \frac{mg(b/a)}{S_1}$

$\Delta_2 = \frac{mg}{S_2}$

$$\Delta = \Delta_1 \cdot \frac{b}{a} + \Delta_2$$

Ques

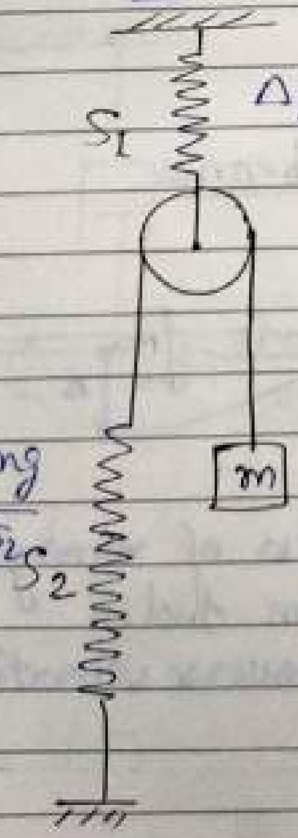


$\Delta_1 = \frac{2mg}{S_1}$

$\Delta = 2\Delta_1$

$\Delta = \frac{4mg}{S_1}$

Ques

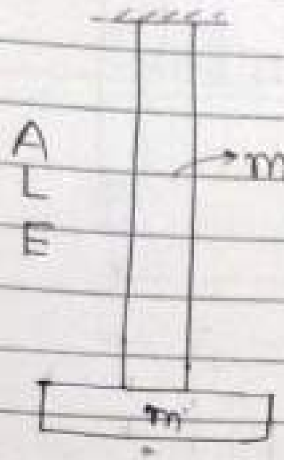


$\Delta_1 = \frac{2mg}{S_1}$

$$\boxed{\Delta = 2\Delta_1 + \Delta_2}$$

$\Delta_2 = \frac{mg}{S_2}$

Longitudinal vibrations of Beams
(along the length)



Longitudinal stiffness

$$S = \frac{AE}{L}$$

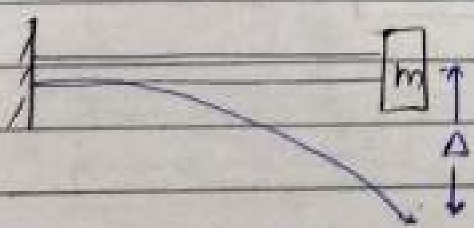
$$\boxed{\omega_n = \sqrt{S/m}}$$

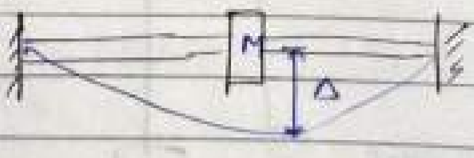
$S_1 = \frac{A_1 E_1}{L_1}$ $\quad S = S_1 + S_2$

$$\boxed{\omega_n = \sqrt{S/m}}$$

$S_2 = \frac{A_2 E_2}{L_2}$

~~Torsi~~ Transverse vibrations of beams
(⊥ to length)





$$\omega_n = \sqrt{\frac{g}{\Delta}} = \sqrt{\frac{S}{m}}$$

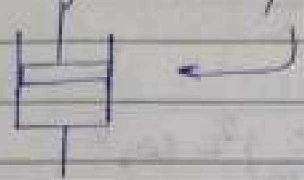
?

Stiffness of ~~spring~~ the system under transverse vibration.

Damped System (Friction $\neq 0$)

Damping $\rightarrow$ Technical name of friction in vibrating system

It is represented by the symbol known as damper.



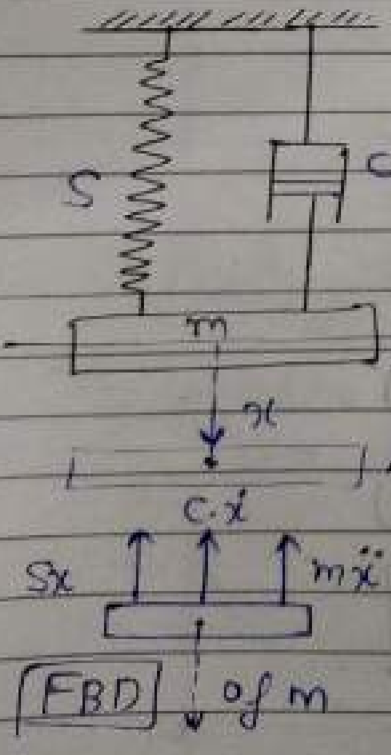
Damping force

Damping force $\propto \dot{x}$

$= c \cdot \dot{x}$

$\hookrightarrow$ coefficient of damping

Damping
- $\rightarrow$ Friction b/w dry surfaces $\longrightarrow$ (coulumb damping (high).
- $\rightarrow$ Viscous damping (low friction)



$m \cdot \ddot{x} + c \cdot \dot{x} + Sx = 0$

$$\boxed{\ddot{x} + \frac{c}{m} \cdot \dot{x} + \frac{S}{m} \cdot x = 0}$$

Equa.^{n} of damped system

The solution of this equa.^{n} will be

$$\boxed{x = A \cdot e^{\alpha_1 t} + B \cdot e^{\alpha_2 t}} \quad (at \; \alpha_1 \neq \alpha_2)$$

or

$$\boxed{x = (A + Bt) \cdot e^{\alpha t}} \quad (\alpha_1 = \alpha_2 = \alpha)$$

A and B are constant

$\alpha_{1,2} \longrightarrow$ Roots of auxilliary equation

$$\boxed{\alpha^2 + \left(\frac{c}{m}\right)\cdot\alpha + \omega_n^2 = 0}$$

$$\alpha_{1,2} = \frac{-c/m \pm \sqrt{(c/m)^2 - 4\omega_n^2}}{2}$$

$$\alpha_{1,2} = \frac{-c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \omega_n^2}$$

Degree of dampness $= \dfrac{(c/2m)^2}{(\omega_n^2)}$

Damping factor or damping ratio $= \zeta = \sqrt{\dfrac{(c/2m)^2}{(\omega_n^2)}}$

$$\zeta = \sqrt{\frac{c^2/4m^2}{s/m}} = \frac{c}{2\sqrt{sm}}$$

$$2\zeta\omega_n \longrightarrow 2 \times \frac{c}{2\sqrt{sm}} \times \sqrt{\frac{s}{m}}$$

$$\longrightarrow c/m$$

The final equa.^{n} of damped system

$$\ddot{x} + 2\zeta\omega_n\cdot\dot{x} + \omega_n^2\cdot x = 0$$

Sol^n.

$$x = A\cdot e^{\alpha_1 t} + B\cdot e^{\alpha_2 t} \quad (\alpha_1 \neq \alpha_2)$$

or

$$x = (A + Bt)\cdot e^{\alpha t} \quad (\alpha_1 = \alpha_2 = \alpha)$$

where, $\boxed{\alpha_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n}$

If Case I – $\xi > 1 \Rightarrow$ over damped system
No. Vibration (coulomb damping)

Case-II

$\xi = 1 \Rightarrow$ Critically damped system
No. Vibration

Case-III

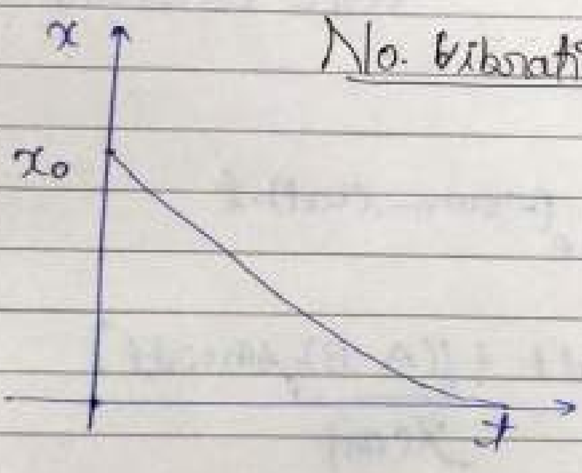
$\xi < 1 \Rightarrow$ Under damped system
Vibration is there (viscous damping)

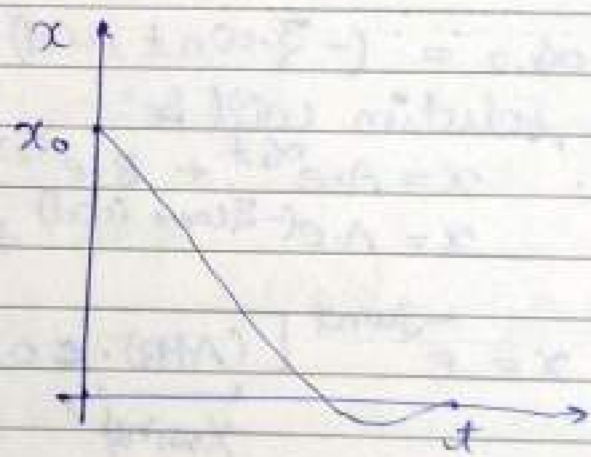
(1) Over damped system ($\xi > 1$)

The solution will be $x = A e^{\alpha_1 t} + B \cdot e^{\alpha_2 t}$

$$\boxed{x = A \cdot e^{(-\xi + \sqrt{\xi^2 - 1})\omega_n t} + B \cdot e^{(-\xi - \sqrt{\xi^2 - 1})\omega_n t}}$$

No. Vibration



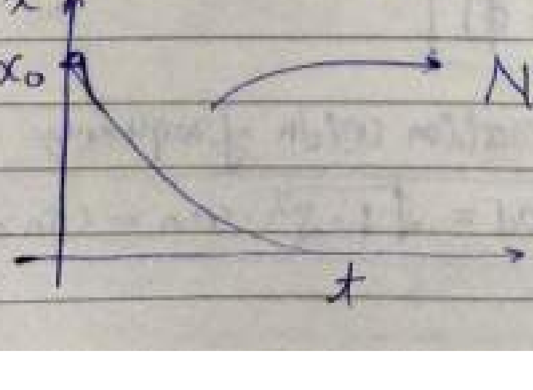


2) Critically damped system ($\xi = 1$)

$\alpha_1 = \alpha_2 = \alpha = -\omega_n$

The solution will be

$$x = (A + Bt) \cdot e^{-\omega_n t}$$

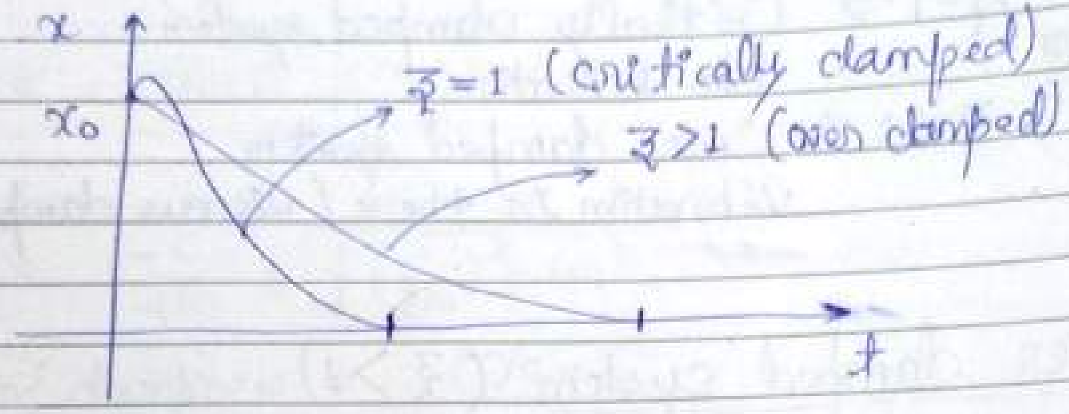


→ No vibration

* Used in rifles
AK-47
AK-56

660/min

(*) Critically damping response is more fast



(3) Underdamped system ($\zeta < 1$)

$$\alpha_{1,2} = -\zeta \omega_n \pm i\underbrace{\left(\sqrt{1-\zeta^2}\right)\cdot \omega_n}_{\omega_d = \text{constant}}$$

$$\alpha_{1,2} = (-\zeta \cdot \omega_n \pm i\omega_d)$$

The solution will be

$$x = A\cdot e^{\alpha_1 t} + B\cdot e^{\alpha_2 t}$$

$$x = A\cdot e^{(-\zeta\omega_n + i\omega_d)} + B\cdot e^{(-\zeta\omega_n - i\omega_d)\cdot t}$$

$$x = e^{-\zeta\omega_n t}\left[\underbrace{(A+B)}_{X\sin\phi}\cdot \cos\omega_d t + \underbrace{i(A-B)}_{X\cos\phi}\sin\omega_d t\right]$$

$$\boxed{x = X e^{-\zeta\omega_n t}\cdot \sin(\omega_d t + \phi)}$$

$X \rightarrow$ constant
$\phi \rightarrow$

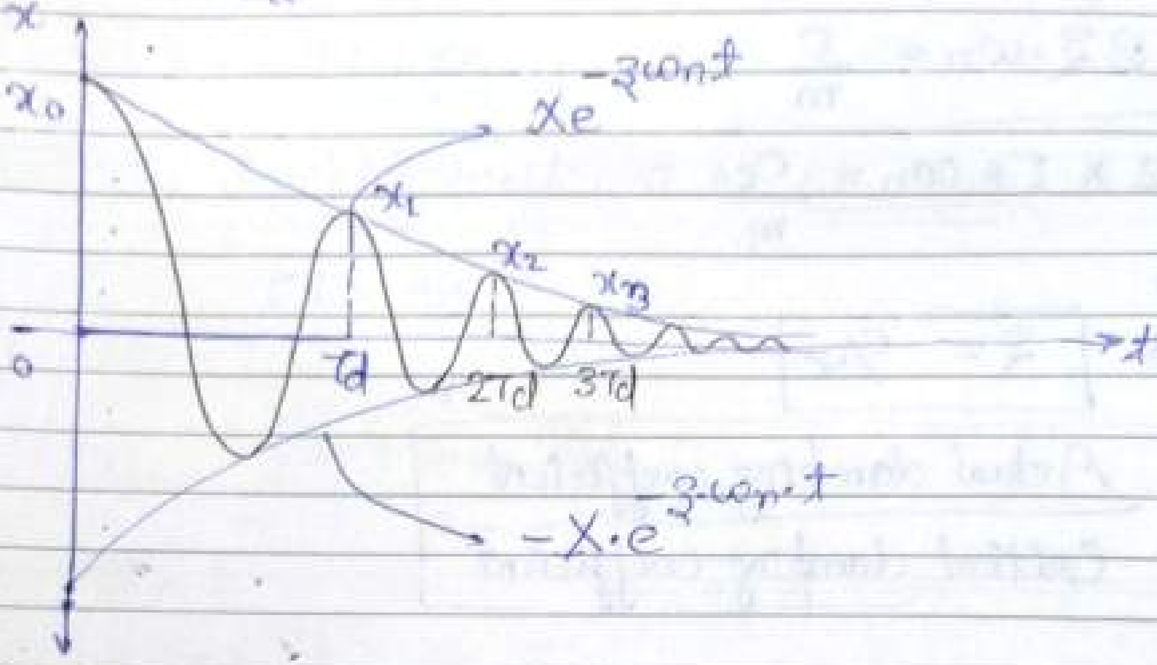
$$\boxed{x = X e^{-\zeta\omega_n \cdot t}\cdot \sin(\omega t + \phi)}$$

Amplitude of vibration decreasing with time

Vibration with frequency

$$\boxed{\omega_d = \sqrt{1-\zeta^2}\cdot \omega_n = \text{const.}}$$

$$T_d = \frac{2\pi}{\omega_d} \Rightarrow \text{constant}$$



Decreament ratio

At $t=0$, $x_0 = X\cdot\sin\phi$

At $t=T_d$, $x_1 = X\cdot e^{-\xi\cdot\omega_n\cdot T_d}\cdot\sin\phi$

At $t=2T_d$, $x_2 = X\cdot e^{-\xi\cdot\omega_n(2T_d)}\cdot\sin\phi$

$$\therefore \frac{x_0}{x_1} = \frac{x_1}{x_2} = \frac{x_2}{x_3} = \dots = e^{\xi\cdot\omega_n T_d} = e^{\delta}$$

Logrithmic decreament (δ)

$$\delta = \ln\left(e^{\xi\omega_n T_d}\right)$$

$$\delta = \xi\omega_n T_d = \frac{\xi\cdot\omega_n\cdot 2\pi}{\omega_d} = \frac{\xi\cdot\omega_n\cdot 2\pi}{\sqrt{1-\xi^2}\times\omega_n}$$

$$\boxed{\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}}}$$