

Steady flow in Pipes :

(a) Flow through a pipe (The Hagen-Poiseuille flow)

Let us consider Laminar flow of incomp. fluid without body forces through a pipe long straight pipe of circular cross section with axial symmetry. Let z be the direction of flow along the axis of the pipe and r denote the radial direction measured outward from the z -axis. The velocity components v_r and v_θ in radial and tangential directions, respectively, are zero.

Under these conditions Eqn of continuity in cylindrical polar coordinates, with $v_r = v_\theta = 0$, also $\frac{\partial}{\partial \theta} = 0$

$$\frac{\partial p}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} (p r v_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (p v_\theta) + \frac{\partial}{\partial z} (p v_z) = 0$$

reduces to Eqn

$$\frac{\partial v_z}{\partial z} = 0$$

i.e $v_z = v_z(r, \theta)$

but due to axial symmetry

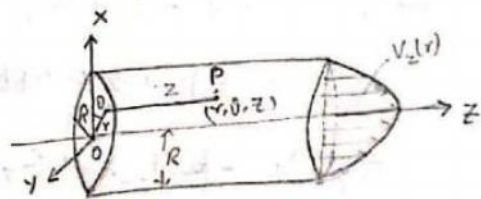
$$v_z = v_z(r) \quad \text{--- (1)}$$

Navier-Stokes Eqn for viscous, incomp. fluid (when viscosity is constant) in cylindrical polar coordinate system are

$$\rho \left(\frac{Dv_r}{Dt} - \frac{v_\theta^2}{r} \right) = \rho X_r - \frac{\partial p}{\partial r} + \mu \left[\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} \right]$$

$$\rho \left(\frac{Dv_\theta}{Dt} - \frac{v_r v_\theta}{r} \right) = \rho X_\theta - \frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[\nabla^2 v_\theta + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} \right]$$

$$\rho \frac{Dv_z}{Dt} = \rho X_z - \frac{\partial p}{\partial z} + \mu \nabla^2 v_z$$



$$v_z = \left(\frac{1}{4\mu} \frac{dp}{dz} \right) r^2 + A \cdot \ln r + B \quad \text{--- (6)}$$

A, B are the constants of integration which are determined by the boundary conditions.

The first boundary condition is found from the symmetry of the flow which requires

$$\text{when } r=0, \quad \frac{dv_z}{dr} = 0$$

The second condition is the no slip condition at the wall

$$\text{when } r=R, \quad v_z = 0$$

Using these conditions we get

$$A = 0$$

$$B = -\frac{1}{4\mu} \frac{dp}{dz} \cdot R^2$$

Substituting these values in (6) we get velocity distribution of the Hagen-Poiseuille flow through a pipe,

$$v_z = \frac{-R^2}{4\mu} \frac{dp}{dz} \cdot \left[1 - \left(\frac{r}{R} \right)^2 \right] \quad \text{--- (7)}$$

which has the form of paraboloid of revolution. The velocity distribution given by (7) can only occur when the flow reaches a fully developed state and laminar. The flow reaches a fully developed state when it travels almost 100 pipe-diameters length from the entrance.

Since $v_r = v_\theta = 0$, $\frac{\partial}{\partial \theta} = 0$ and $X_r = X_\theta = X_z = 0$
 So above Eqⁿ reduces to

$$\frac{\partial p}{\partial r} = 0 \quad \text{--- (2)}$$

$$\frac{\partial p}{\partial \theta} = 0 \quad \text{--- (3)}$$

and $f \cdot v_z \cdot \frac{\partial v_z}{\partial z} = -\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \right) v_z$

Eqⁿ (2) and (3) shows that $p = p(z)$. Since v_z is a fⁿ of r only ⁽⁴⁾
 therefore Eqⁿ (4) reduces into

$$0 = -\frac{dp}{dz} + \mu \left(\frac{d^2 v_z}{dr^2} + \frac{1}{r} \frac{dv_z}{dr} \right)$$

i.e $\mu \left(\frac{d^2 v_z}{dr^2} + \frac{1}{r} \frac{dv_z}{dr} \right) = \frac{dp}{dz} \quad \text{--- (5)}$

This Eqⁿ is the balance of the shearing force and pressure force so that there is no resultant force in the direction of z .

Again Eqⁿ (5) shows that $\frac{dp}{dz}$ is constant.
 Under this condition SIⁿ of Eqⁿ (5) is

$$r \cdot \frac{d^2 v_z}{dr^2} + \frac{dv_z}{dr} = \frac{1}{\mu} \frac{dp}{dz} \cdot r$$

$$\Rightarrow \frac{d}{dr} \left(r \cdot \frac{dv_z}{dr} \right) = \frac{1}{\mu} \cdot \frac{dp}{dz} \cdot r$$

$$\Rightarrow r \frac{dv_z}{dr} = \frac{1}{\mu} \frac{dp}{dz} \frac{r^2}{2} + A$$

$$\Rightarrow \frac{dv_z}{dr} = \frac{1}{\mu} \frac{dp}{dz} \cdot \frac{r}{2} + \frac{A}{r}$$

$$v_z = \frac{1}{2\mu} \frac{dp}{dz} \frac{r^2}{4} + A \log r + B$$

(4)

The maximum and Average velocities:

It is clearly from Eqn (7) that maxi. velocity occurs at the centre of the pipe where $r=0$.

$$(V_z)_{\max} = -\frac{R^2}{4\mu} \left(\frac{dp}{dz} \right)$$

where $\frac{dp}{dz} < 0$.

(10) The average velocity $V_{z(av)}$ can be obtained from

$$V_{z(av)} = \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R V_z r dr d\theta$$

$$= \frac{1}{\pi R^2} 2\pi \int_0^R V_z \cdot r dr$$

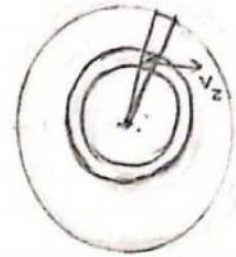
$$= \frac{2}{R^2} \left(-\frac{R^2}{4\mu} \frac{dp}{dz} \right) \int_0^R r \cdot \left\{ 1 - \frac{r^2}{R^2} \right\} dr$$

$$= -\frac{1}{2\mu} \frac{dp}{dz} \left(\frac{r^2}{2} - \frac{r^4}{4R^2} \right)_0^R$$

$$= -\frac{1}{2\mu} \frac{dp}{dz} \cdot \left(\frac{R^2}{2} - \frac{R^2}{4} \right)$$

$$V_{z(av)} = -\frac{R^2}{8\mu} \frac{dp}{dz}$$

$$V_{z(av)} = \frac{(V_z)_{\max}}{2}$$



The Volumetric flow

Q = flow of volume in unit time through cross-section

$$= \int_0^{2\pi} \int_0^R V_z \cdot r \, d\theta \, dr$$

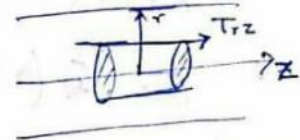
$$= \pi R^2 \cdot V_{avg}$$

$$Q = \frac{\pi R^4}{8\mu} \left(-\frac{dp}{dz} \right)$$

∴ mass flux per unit time passing any cross-section $M = \rho \frac{\pi R^4}{8\mu} \left(-\frac{dp}{dz} \right)$
 This result is Poiseuille's law. we can calculate μ .

(iii) The shearing stress at the wall:

shearing stress at any point is $\tau_{rz} = +\mu \left(\frac{\partial V_z}{\partial r} + \frac{\partial V_r}{\partial z} \right)$



$$= +\mu \frac{\partial V_z}{\partial r}$$

$$\tau_{rz} = +\mu \frac{dV_z}{dr} = \left(\frac{+dp}{dz} \right) \frac{r}{2} \quad \text{using (7)}$$

at the wall

$$\text{Shearing stress} = -(\tau_{rz})_{r=R} = +\mu \cdot \frac{R}{2\mu} \frac{dp}{dz}$$

$$(\tau_{rz})_{r=R} = +\frac{R}{2} \frac{dp}{dz}$$

$$= +4\mu \frac{V_{avg}}{R}$$

