Unit - I 4/7½

FkeâeF - I

2. (a) Prove that the set of all positive rational numbers, under the composition '$*$' defined by $a * b = \frac{ab}{2}$ form an infinite abelian group.

emeæ keâeepeS ekeâ Oevee lcekeâ heej ceÙe meKÙeeDee keâe meceÛÛe, $a * b = \frac{ab}{2}$ Éeje heejYeeef<ele mee›eâÙee '$*$' keâ meehe#e Skeâ Devevle ›eâceeJeeveÙece mecen keâe eveecele keâjlee n~

(b) If H and K are two subgroups of a group G, then prove that HK is a subgroup of G if and only if HK = KH.

Ùeeo H leLee K mecen G keâ oe Ghemecen ne lee oMeeFÙe ekeâ HK, G keâe Skeâ Ghemecen neiee Ùeeo Deej keâJeue Ùeeo HK = KH ne~

3. (a) Prove that the order of each subgroup of a finite group is a divisor of the order of the group. Also, show that its converse is not true.

emeæ keâeepeS ekeâ Skeâ heejecele mecen keâ øelÙekeâ Ghemecen keâe keâeeš Gmekeâe keâeeš keâe Yeepekeâ nelee n~

A (Printed Pages 8)

Roll No. ____________

S-669

B.A./B.Sc. (Part-I) Examination, 2015

(Regular)

MATHEMATICS

First Paper

(Algebra)

Time Allowed : Three Hours] [Maximum Marks : B.A. : 25, B.Sc. : 50

Note : Answer Question No. 1 and four more questions, selecting one question from each Unit.

øeMve me.1 keâe Gòej oeepeS leLee øelÙekeâ FkeâeF me Skeâ øeMve Ûevele nS, Ùeej DevÙe øeMvee keâ Gòej oeepeS~

1. Attempt all parts : 10/20

meYee Yeeie nue keâeepeS :

(a) Let $a, b, c, d \in \mathbb{Z}$ and n be a fixed positive integer. If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then show that :

ceeve ueeepeS ekeâ $a, b, c, d \in \mathbb{Z}$ leLee n Skeâ eveÙele Oeveelcekeâ heCeekeâ n~ Ùeeo $a \equiv b \pmod{n}$ leLee $c \equiv d \pmod{n}$ ne lee oMeeFÙe ekeâ

(i) $a + c \equiv b + d \pmod{n}$

(ii) $ac \equiv bd \pmod{n}$.

(b) Let (G, $*$) be a group. Then prove that :

ceeve ueeepeS (G, $*$) Skeâ mecen n~ lee emeæ keâeepeÙe ekeâ :

$(a * b)^{-1} = b^{-1} * a^{-1} \; \forall \, a, b \in G.$

(c) Define the order of an element of a group. Find the order of each element of a multiplicative group $\{1, w, w^2\}$. Is this group cyclic?

mecen keâ DeJeÙeJe keâe keâeeš keâe heejYee<ee ieeCele mecen $\{1, w, w^2\}$ keâ ØelÙekeâ DeJeÙeJe keâe keâeeš %eele keâeepeS~ keÙee Ùen mecen Ûe›eâeÙe n?

(d) Find the remainder when 9^{107} is divided by 11, using Fermat's theorem.

Heâjcee (Fermat) keâe ØeceeÙe keâe ØeÙeeie keâjle nÙe DeJeMe<e %eele keâeepeS peye 9^{107} keâe 11 me Yeeie eoÙee peelee n~

(e) Find the inverse of the permutation :

evecve ›eâceÛeÙe keâe Øeeleueece %eele keâeepeS :

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$$

(f) Define external direct product of two groups.

oe mecene keâ yee¢eo° ieCeve keâe heejYeee<ele keâeepeS~

(g) If R is a ring such that $a^2 = a \; \forall \, a \in R$, then show that each element of R has its own additive inverse.

Ùeeo JeueÙe R Fme Øekeâej me n ekeâ $a^2 = a \; \forall \, a \in R$, lee oMeeFÙe ekeâ JeueÙe keâe ØelÙekeâ DeJeÙeJe mJeÙe Ùeepe Øeeleueece n~

(h) Prove that the multiplicative inverse of a non-zero element of a field is unique.

emeæ keâeepeS ekeâ Skeâ #e$e keâ Skeâ DeMevÙe DeJeÙeJe keâe ieCepe Øeeleueece DeeÉleeÙe nelee n~

(i) Prove that the intersection of two subspaces of a vector space V(F) is also a subspace of V(F).

emeæ keâeepeS ekeâ Skeâ meeoMe mecee° V(F) keâ oe Ghemecee° Ùee keâe ØeeleÛÚove Yee V(F) keâe Skeâ Ghemecee° nelee n~

(j) Show that the vectors (1, 2,1), (2,1,0), (1,–1, 2) form a basis of R^3.

oMeeFS ekeâ meeoMe (1, 2,1), (2,1,0), (1,–1, 2) R^3 keâe Skeâ DeeOeej eveecele keâjle n~

(i) if $a, b \in F$ and α is a non zero element of V.

Ùeeo $a, b \in F$ leLee α, V keâe Skeâ DeMevÙe DeJeÙeJe ne lee

$a\alpha = b\alpha \Rightarrow a = b$.

(ii) if $\alpha, \beta \in V$ and a is a non-zero element of F,

Ùeeo $\alpha, \beta \in V$ leLee a, F keâe Skeâ DeMevÙe DeJeÙeJe ne lee,

$a\alpha = a\beta \Rightarrow \alpha = \beta$.

(b) If V is a finite dimensional vector space, then show that any two bases of V have the same number of elements.

Ùeeo V Skeâ heejecele eJeceeÙe meeoMe mecee° ne lee oMeeFÙe ekeâ V keâ keâeF oe DeeOeeje ce DeJeÙeJee keâe meKÙee meceeve nelee n~

9. If w_1 and w_2 are finite-dimensional subspaces of vector space V, then prove that :

Ùeeo w_1 leLee w_2 meeoMe mecee° V keâe heejecele-eJeceeÙe Ghe mecee°Ùee ne lee emeæ keâeepeÙe ekeâ :

$\dim (w_1 + w_2) = \dim w_1 + \dim w_2 - \dim (w_1 \cap w_2)$.

(b) If H is a normal subgroup of a group G, then prove that the quotient set G/H of all the right cosets of H in G form a group under the composition defined by (Ha) (Hb) = Hab.

Ùeeo H ekeâmee mecen G keâe Skeâ ØemeeceevÙe Ghemecen ne lee emeæ keâeepeS ekeâ G ce H keâ meYee oe#eCe menmeceÛÛee keâe eJeYeeie meceÛÛe G/H, (Ha) (Hb) = Hab Éeje heejYeee<ele mee›eâÙee keâ meehe#e Skeâ mecen eveecele keâjlee n~

Unit - II 4/7½

FkeâeF - II

4. (a) State and prove Cayley Theorem.

keâue keâe ØeceÙe keâe keâLeve keâjle nÙe emeæ keâeepeS~

(b) If f is a homomorphism of a group G into a group G' with Kernel K, then prove that K is a normal subgroup of G.

Ùeeo f ekeâmee mecen G keâe G' ce Skeâ meceekeâeejlee n epemekeâe Dee° K ne lee emeæ keâeepeS ekeâ K, G keâe Skeâ ØemeeceevÙe Ghemecen nelee n~

5. (a) Let H be a normal subgroup of a group G and $f : G \to G/H$ be a map defined by $f(x) = Hx \ \forall x \in G$. Then prove that f is a homomorphism of G onto G/H with H as a Kernel of f.

ceeve ueeepeÙe ekeâ H ekeâmee mecen G keâe ØemeeceevÙe Ghemecen n leLee $f : G \rightarrow G/H$, Skeâ Øeeleeeûe$eCe n pee $f(x) = Hx \;\forall\, x \in G$ Éeje heejYeeef<ele n~ emeæ keâeepeS ekeâ f, G keâe G/H hej DeeÛÚeokeâ meceekeâeejlee n epemekeâe Dee° H n~

(b) Define internal direct product of the sub-groups H_i, $i = 1, 2, ..., n$ of a group G. Prove that if G is the internal direct product of subgroups $H_1, H_2, .., H_n$, then each $g \in G$ can be uniquely written as $g = h_1, h_2 ... h_n$ where $h_i \in H_i$ $(i = 1, ..., n)$.

Skeâ mecen G keâ Ghemecene H_i, $i = 1, 2, ..., n$ keâ Deevleejkeâ Deveueece ieCeve keâe heejYeeef<ele emeæ keâeepeS ekeâ Ùeeo G Ghemecene $H_1, ..., H_n$ keâe Deevleejkeâ Deveueece ieCeve ne lee ØelÙekeâ $g \in G$ keâe DeeÉleeÙe ¤he $g = h_1, h_2 ... h_n$ ce eueKee pee mekeâlee n pene $h_i \in H_i$ $(i = 1, ..., n)$.

Unit - III 4/7½

FkeâeF - III

6. (a) Prove that a non-empty subset S of a ring R is a subring of R if and only if :

emeæ keâeepeS ekeâ JeueÙe R keâe Skeâ Deejòeâ GhemeceÛÛe S, R keâe Skeâ GheJeueÙe leYee Deej leYee neiee peye :

(i) $a, b \in S \Rightarrow a - b \in S$

(ii) $a, b \in S \Rightarrow a\, b \in S$

(b) Prove that every finite integral domain is a field.

emeæ keâeepeS ØelÙekeâ heejecele heCeeekeâle Skeâ #e$e nelee n~

7. (a) Let F be a field. Then show that the set $M_2(F)$ of all 2×2 matrices over F forms a ring under matrix addition and multiplication. Does $M_2(F)$ have zero divisors?

ceevee F Skeâ #e$e n~ lee oMeeFÙe ekeâ $M_2(F)$ pee F hej meYee 2×2 DeeJÙene keâe meceÛÛe n, DeeJÙen Ùeeie SJe ieCeve keâ meehe#e Skeâ JeueÙe eveecele keâjlee ~ keÙee $M_2(F)$ keâ MevÙe Yeepekeâ n?

(b) Let ϕ be a homomorphism of a ring R into a ring R' with Kernel K. Then show that $\phi(R)$ is a subring of R' and R/K is isomorphic to ϕ (R).

ceeve ueeepeS ekeâ ϕ JeueÙe R me JeueÙe R' ce Dee° K keâ meeLe Skeâ meceekeâeejlee n~ lee oMeeFÙe ekeâ $\phi(R)$, R' keâe Skeâ GheJeueÙe n leLee R/K, ϕ (R) keâ leuÙeekeâej n~

Unit - IV 3/7½

FkeâeF - IV

8. (a) In a vector space V(F) prove that :

Skeâ meeoMe mecee° V(F) ce emeæ keâeepeÙe :